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On the
One-Dimensional Magnetohydrodynamic Flow
in an Annulus

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Abstract

Inviscid flow in the circumferential direction of a liquid conductor confined between two concentric cylindrical electrodes and driven by the Lorentz force created by crossed electric and magnetic fields is considered. It is shown that, in general, no well-behaved solutions exist to the inviscid problem without secondary flow. Two special solutions are given corresponding to zero field and zero current degeneracies of the governing equations and boundary conditions. It is also shown that hydrostatic equilibrium is incompatible with current-flow, even when no external magnetic fields are applied. An additional boundary condition on the radial magnetic field is also derived as a consequence of the basic formulation.

A. Introduction and Formulation of the Problem

We consider the problem of a liquid conductor (such as mercury) confined between two concentric cylindrical electrodes. The liquid metal is electromagnetically accelerated by the Lorentz force in the circumferential direction created when a current is passed between the two electrodes (held at different potentials) and an axial magnetic field is applied. See the Figure. The governing equations are the coupled momentum and Maxwell equations together with Ohm's law in the special case of steady state, inviscid flow with constant properties. Axial symmetry is also assumed. Gravity body forces have been neglected. The equations in vector form are then:

$$\nabla \cdot \underline{v} = 0 \qquad \text{(conservation of mass)} \qquad (1)$$

$$\nabla \cdot \underline{B} = 0 \qquad \text{(conservation of magnetic flux)} \qquad (2)$$

$$\nabla \times \underline{B} = \mu \underline{I} \qquad \text{(Ampere's law)} \qquad (3)$$

$$\nabla \times \underline{E} = 0 \qquad \text{(Faraday's law)} \qquad (4)$$

$$\underline{I} = \sigma(\underline{E} + \underline{v} \times \underline{B}) \qquad \text{(Ohm's law)} \qquad (5)$$

$$\nabla v^2/2 - \underline{v} \times (\nabla \times \underline{v}) = -\nabla p/\rho + (1/\rho)\,(\underline{I} \times B) \qquad \text{(conservation of momentum)} \qquad (6)$$

Combining Eqs. (3) and (6) gives the "momentum eq.", and (3), (4), (5) gives the "magnetic eq.". Thus, we have:

$$\nabla \cdot \underline{v} = 0 \qquad (7)$$

$$\nabla \cdot \underline{B} = 0 \qquad (8)$$

$$\nabla \times (\underline{B} \times \underline{v}) + (1/\mu\,\sigma)\,\nabla \times (\nabla \times \underline{B}) = 0 \qquad \text{(magnetic equation)} \qquad (9)$$

$$\nabla v^2/2 - \underline{v} \times (\nabla \times \underline{v}) = -\nabla p/\rho + (1/\rho\mu)\,(\nabla \times \underline{B}) \times \underline{B} \qquad \text{(momentum equation)} \qquad (10)$$

with the boundary conditions:*

(1) $\phi_o = - \int_{r_1}^{r_2} [(1/\mu\sigma) \cdot (\partial B_\theta/\partial z) + v_\theta B_z] dr$ for all z

(2) $B_\theta = 0$ at $z = L$, $B_\theta = \mu I/2\pi r$ at $z = -L$, $I \neq 0$

(3) $E_z = 0$ at $r = r_1, r_2 \Rightarrow v_\theta B_r = -(1/\mu\sigma r)\, \partial/\partial r\, (rB_\theta)$ at $r = r_1, r_2$

(4) $E_\theta = 0$ at $r = r_1, r_2 \Rightarrow \partial B_r/\partial z - \partial B_z/\partial r = 0$ at $r = r_1, r_2$

where $\underline{B}$ is the magnetic induction, $\underline{v}$ is the velocity of the fluid, μ is the magnetic permeability, σ is the electrical conductivity, p is the pressure, ϕ_o is the potential difference between the two electrodes. We have also assumed that there is no secondary flow.

The boundary conditions are consequences of the following physical considerations. Boundary condition (1) is a statement that the difference in potential between the inner and outer electrodes is ϕ_o, which is a line integral of $\underline{E}$ from one electrode to the other and independent of the path by virtue of Eq. (4). We have chosen a radial path for the line integral and $\underline{E}$ has been replaced by its value through Ohm's law and $\underline{J}$ has been replaced by curl $\underline{B}$ through Ampere's law. Boundary condition (2) is found from the integral form of Ampere's law and states that all of the current comes in from the bottom and none leaves through the top. See the Figure. This is consistent with insulated end plates, $J_z = 0$ at $z = \pm L$. Assuming that the electrodes are perfect conductors, $\underline{E}$ must be radial at the electrode surfaces.

*Boundary conditions on B_r and B_z are also required for a unique determination of the fields. However, these boundary conditions are dependent on the geometry of the magnet producing the external field. An internal boundary condition on B_r is derived in Section D.

Then Ohm's law together with $\underline{v} = v_\theta \underline{1}_\theta$ only* and $\underline{J}$ replaced by curl $\underline{B}$ yields boundary conditions (3) and (4). For arbitrary values of the physical parameters, the equations split up into autonomous subsystems, i.e., the magnetic field may be solved for independently of the flow and the coupling between the magnetic and velocity fields enters through the boundary conditions. We prove here a nonexistence theorem: no well-behaved solutions of the above equations and boundary conditions exist.

In scalar, component form, the equations become

$$\partial v_\theta/\partial\theta = 0, \text{ conservation of mass } (\underline{v} = \underline{1}_\theta v_\theta \text{ only because secondary flow is neglected}) \tag{11}$$

$$(1/r)\, \partial/\partial r\, (rB_r) + \partial B_z/\partial z = 0, \quad \text{conservation of magnetic flux} \tag{12}$$

$$\partial/\partial z\, (\partial B_r/\partial z - \partial B_z/\partial r) = 0, \quad r \text{ - magnetic eq.} \tag{13}$$

$$\partial/\partial r\, [\, r(\partial B_z/\partial r - \partial B_r/\partial z)\,] = 0, \quad z \text{ - magnetic eq.} \tag{14}$$

$$-\partial^2 B_\theta/\partial z^2 - \partial/\partial r[(1/r)\, \partial/\partial r\, (rB_\theta)\,] = \mu\sigma\, [\partial/\partial z\, (v_\theta B_z) + \partial/\partial r\, (v_\theta B_r)\,] \tag{15}$$

θ - magnetic eq.

$$-v_\theta^2/r = -(1/\rho)\, \partial p/\partial r + (1/\rho\mu)\, [\, B_z\, (\partial B_r/\partial z - \partial B_z/\partial r) - (B_\theta/r)\, \partial/\partial r\, (rB_\theta)\,] \tag{16}$$

r - momentum eq.

$$B_z \partial B_\theta/\partial z + (B_r/r)\, \partial/\partial r\, (rB_\theta) = 0, \quad \theta \text{ - momentum eq.} \tag{17}$$

*The assumption that v_θ is the only velocity component is consistent with the assumption that only viscous forces may give rise to secondary flow. We will show, however, that this leads to a possible contradiction.

$$0 = (1/\rho)\, \partial p/\partial z + (1/\rho\mu)\, [B_\theta \partial B_\theta/\partial z + B_r \partial B_r/\partial z -$$

$$B_r \partial B_z/\partial r]\text{, z - momentum eq.} \qquad (18)$$

with boundary conditions (1) - (4).

Eqs. (13) and (14) with boundary condition (4) yield

$$\partial B_z/\partial r - \partial B_r/\partial z = 0\text{, r \& z magnetic eq.} \qquad (19)$$

This proves that $J_\theta = 0$ (Eq. (19) is the θ component of curl $\underline{B}$). We observe that this result is independent of the form of the momentum equation and, therefore, is valid for viscous flow as well. Thus the assumption that J_θ = 0 often made[1,2,3] is here proved as a consequence of the boundary condition that $E_\theta = 0$ at the electrodes (which merely assumes that the electrodes are perfect conductors) and Eqs. (3, 4, 5).

Eq. (19) replaces Eqs. (13), (14) and boundary condition (4). Note that Eqs. (19) and (12) form an autonomous subsystem for B_r and B_z, and the results may then be put into Eq. (17) for B_θ.

B. Nonexistence Proof

We shall now prove nonexistence of solutions to the problem formulated above for the two cases $v_\theta \neq 0$, and $v_\theta = 0$.

1. $v_\theta \neq 0$

From boundary condition (1)

$$-v_\theta B_z = f_1(r) + \frac{1}{\mu\sigma}\frac{\partial B_\theta}{\partial z} \qquad (20)$$

where f_1 is restricted by $\phi_o = \int_{r_1}^{r_2} f_1(r)\, dr$. By virtue of Eq. (20), the z-derivatives in Eq. (15) sum to zero, leaving

$$-v_\theta B_r = g_1(z) + \frac{1}{\mu\sigma r} \frac{\partial}{\partial r} (rB_\theta) \tag{21}$$

From boundary condition (3), $g_1(z) = 0$. From Eqs. (21) and (22)

$$\frac{B_r}{B_z} = \frac{\frac{1}{\mu\sigma r} \frac{\partial}{\partial r} (rB_\theta)}{f_1(r) + \frac{1}{\mu\sigma r} \frac{\partial}{\partial z} (rB_\theta)} \tag{23}$$

Using Eq. (23) to eliminate B_z/B_r in Eq. (17) and making the substitution

$$rB_\theta = F(r, z) \tag{24}$$

we find

$$\left.\begin{aligned} &\left(\frac{\partial F}{\partial r}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2 + \mu\sigma r\, f_1(r) \frac{\partial F}{\partial z} = 0 \\ &\text{with boundary condition (2)} \\ &F(z = -L) = \mu I/2\pi\ ,\ F(z = +L) = 0\,. \end{aligned}\right\} \tag{25}$$

We also have at $z = \pm L$ that $\frac{\partial F}{\partial r} = 0$ and $\frac{\partial F}{\partial z} = 0$. The former condition is just the r-derivative of the boundary conditions of (25) and the latter condition follows from applying the former to Eq. (17) at $z = \pm L$, provided that $B_z \neq 0$ at $z = \pm L$. If $B_z = 0$, then we obtain a special solution given in

Section C.

By successive differentiation of Eq. (25) with respect to z and application of the boundary conditions on the planes $z = \pm L$, it is easily seen that all z derivatives of F vanish on $z = \pm L$. Since F is clearly a function of z by boundary condition (2), this proves that no well-behaved* solutions exist to Eq. (25). Consequently, the formulation with the assumptions of inviscid flow and one velocity component admits no well-behaved solutions.

2. $v_\theta = 0$

In this case, boundary condition (1) gives

$$\partial B_\theta / \partial z = f_4(r) \tag{26}$$

and integration of Eq. (15) gives

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = g_3(z). \tag{27}$$

Integration of Eqs. (26) and (27) and application of boundary conditions (2) and (3) (for $v_\theta = 0$) yield the unique solution

$$B_\theta = \frac{\mu I}{4\pi r}\left[-\frac{z}{L} + 1\right] \tag{28}$$

The current-voltage characteristic obtained from boundary condition (1) is the same as for the zero field solution found in Section C below:

*A function may be nonzero and all its derivatives may vanish at a boundary if it exhibits exponential behavior, e.g., $f(x) \to e^{-1/x}$ as $x \to 0^+$. We have excluded this type of singular behavior by use of the terminology "well-behaved."

$$\phi_o = \frac{I}{4\pi\sigma L} \ln \frac{r_2}{r_1} \tag{29}$$

B_r and B_z are obtained from the solution of the autonomous subsystem, Eqs. (12) and (19). To determine these fields uniquely requires certain external boundary conditions, in addition to the internal boundary condition derived in Section D. The pressure is found directly from Eqs. (16) and (18) giving in turn $p = p(z)$ and

$$p(z) = -\frac{B_\theta^2}{2\mu} + f_5(r) \tag{30}$$

where B_θ is given above by Eq. (28). This is a contradiction because no combination of $p(z) - f_5(r)$ can give the behavior of B_θ^2, regardless of B_r and B_z.

Eqs. (28) and (17) yield $B_z = 0$ and then Eqs. (12) and (19) show $B_r = c/r$. We have shown that the only configuration compatible with no flow is $I = 0$ and $\phi_o = 0$, for which all components of magnetic induction are zero, p = constant and velocity is zero. This is the trivial solution. The contradiction indicates that the z-body force $J_r B_\theta$ cannot be balanced by a pressure gradient. This suggests that some kind of cellular flow pattern must be created by the passage of current.

C. Special Solutions

Two special solutions are of interest because they demonstrate that by a certain degeneracy in the equations or boundary conditions, solutions may exist. The special solutions are (1) $B_z = 0$ and (2) $I = 0$. These may be thought of as the "zero field solution" and the "zero current solution," respectively.

1. $B_z = 0$

From Eqs. (12) and (19), we obtain

$$B_r = c_1/r \tag{31}$$

Boundary condition (1) yields $\partial B_\theta/\partial z = f(r)$. Integrating with respect to z and using boundary condition (2) gives for B_θ:

$$B_\theta = \frac{\mu I}{4\pi r}\left[-\frac{z}{L} + 1\right] \tag{32}$$

Substituting Eq. (32) into boundary condition (1) gives the current-voltage characteristic of the device:

$$\phi_o = \frac{I}{4\pi\sigma L} \ln \frac{r_2}{r_1} \tag{33}$$

Now from boundary condition (3), $v_\theta B_r = 0$ at $r = r_1$, $r = r_2$. From Eq. (31), $v_\theta c_1/r = 0$ at $r = r_1$, $r = r_2$. Two cases are possible:

(a) $v_\theta = 0$ on $r = r_1$, $r = r_2$.

From Eq. (15), $\frac{\partial}{\partial r}(v_\theta B_r) = 0$ or $v_\theta B_r = v_\theta c_1/r = g(z)$. Since $v_\theta = \frac{rg(z)}{c_1}$, then $v_\theta = 0$ on $r = r_1$, $r = r_2$ is possible only if $g(z) = 0$. Thus $v_\theta = 0$ everywhere. This leads to a contradiction as shown above in Section B.2.

(b) $B_r = 0$ on $r = r_1$, $r = r_2$.

This requires $c_1 = 0$ so $B_r = 0$ everywhere. From Eq. (18),

$$p + \frac{B_\theta^2}{2\mu} = f_2(r) \tag{34}$$

where B_θ is known from Eq. (32). Eq. (16) gives v_θ :

$$v_\theta^2 = \frac{r}{\rho} \frac{\partial p}{\partial r} = \frac{r}{\rho} \left[f_2' (r) - \frac{1}{2\mu} \frac{\partial B_\theta^2}{\partial r} \right] \tag{35}$$

Note that the velocity distribution is given in terms of a single arbitrary function of r, which is to be expected for an inviscid flow.

2. $I = 0$

Boundary condition (2) now states $B_\theta = 0$ at $z = \pm L$. Now B_θ is created by the current flowing in the central electrode (through Ampere's law). Thus if the current is zero, so also must be B_θ. From boundary condition (3), $v_\theta B_r = 0$ at $r = r_1$, $r = r_2$. Two possibilities follow.

(a) $v_\theta = 0$ on $r = r_1$, $r = r_2$.

From boundary condition (1) $v_\theta B_z = f_1(r)$. Then Eq. (15) gives $v_\theta B_r = g_2(z)$. $g_2(z)$ must equal zero by virtue of boundary condition (3). Since in this subcase, B_r is not to be taken as zero, this implies $v_\theta = 0$ everywhere which is the trivial solution.

(b) $B_r = 0$ on $r = r_1$, $r = r_2$.

Eliminating B_z from Eqs. (12) and (19) yields for B_r

$$\frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} (rB_r) \right] + \frac{\partial^2 B_r}{\partial z^2} = 0 \tag{36}$$

subject to the boundary conditions $B_r = 0$ on $z = \pm L$ (see Section D for a derivation of this boundary condition) and $r = r_1$, r_2. By separation of variables for Eq. (36), it is easily shown that $B_r \equiv 0$ is the only solution satisfying the boundary conditions. From

Eqs. (12) and (19), B_z = constant = B_o. Eq. (15) yields $v_\theta = f_3(r)$ and Eq. (18) gives $p = p(r)$. Then Eq. (16) relates $p(r)$ to $v_\theta(r)$,

$$\partial p/\partial r = \rho v_\theta^2/r = \rho f_3^2(r)/r. \tag{37}$$

We observe that the velocity field is again given by an arbitrary function of r.

The current-voltage characteristic (with I = 0, $B_\theta = 0$) is then deduced from boundary condition (1) as

$$\phi_o = - B_o \int_{r_1}^{r_2} f(r)dr \tag{38}$$

which merely states that, if no current flows, the "applied emf" is equal and opposite to the "back emf."

D. <u>Derived Boundary Condition, $B_r = 0$ on $z = \pm L$</u>

From boundary condition (1),

$$\frac{1}{\mu\sigma}\frac{\partial B_\theta}{\partial z} + v_\theta B_z = f_1(r). \tag{39}$$

Thus the two z-derivative terms in Eq. (15) cancel out and integrating the remainder of Eq. (15) gives

$$-\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = \mu\sigma v_\theta B_r - g_3(z) \tag{40}$$

From boundary condition (2), at $z = \pm L$, $\frac{\partial}{\partial r}(rB_\theta) = 0$. Thus $g_3(z = \pm L) = \mu\sigma v_\theta B_r$ evaluated at $z = \pm L$. Therefore, on the planes $z = \pm L$, $v_\theta B_r$ is a constant (independent of r). We can evaluate this constant at the corners (r_1, L), (r_2, L), $(r_1, -L)$, and $(r_2, -L)$ provided that $v_\theta B_r$ is a continuous function of its arguments. From boundary condition (3),

$$-\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = \mu\sigma v_\theta B_r \quad \text{at} \quad r = r_1,\ r = r_2.$$

By comparison with Eq. (40), $g_3(z) = 0$ at the corners and, in fact, $g_3(z) = 0$ everywhere. Thus

$$v_\theta B_r = 0 \text{ on } z = \pm L \tag{41}$$

We treat each of the two possibilities in turn.

(a) $v_\theta = 0$ on $z = \pm L$

Boundary condition (1) applied at $z = \pm L$ gives $\frac{\partial B_\theta}{\partial z} = f_4(r) \neq 0$ if $\phi_o \neq 0$. Now Eq. (17) evaluated at $z = \pm L$ (using boundary condition (2)) gives $B_z\, \partial B_\theta/\partial z = 0$. If $B_z(z = \pm L) \neq 0$ ($B_z = 0$ yields a special solution) then $\partial B_\theta/\partial z = 0$ on $z = \pm L$. This contradicts the result from boundary condition (1)where $v_\theta = 0$ on $z = \pm L$. Thus v_θ cannot equal zero on $z = \pm L$.

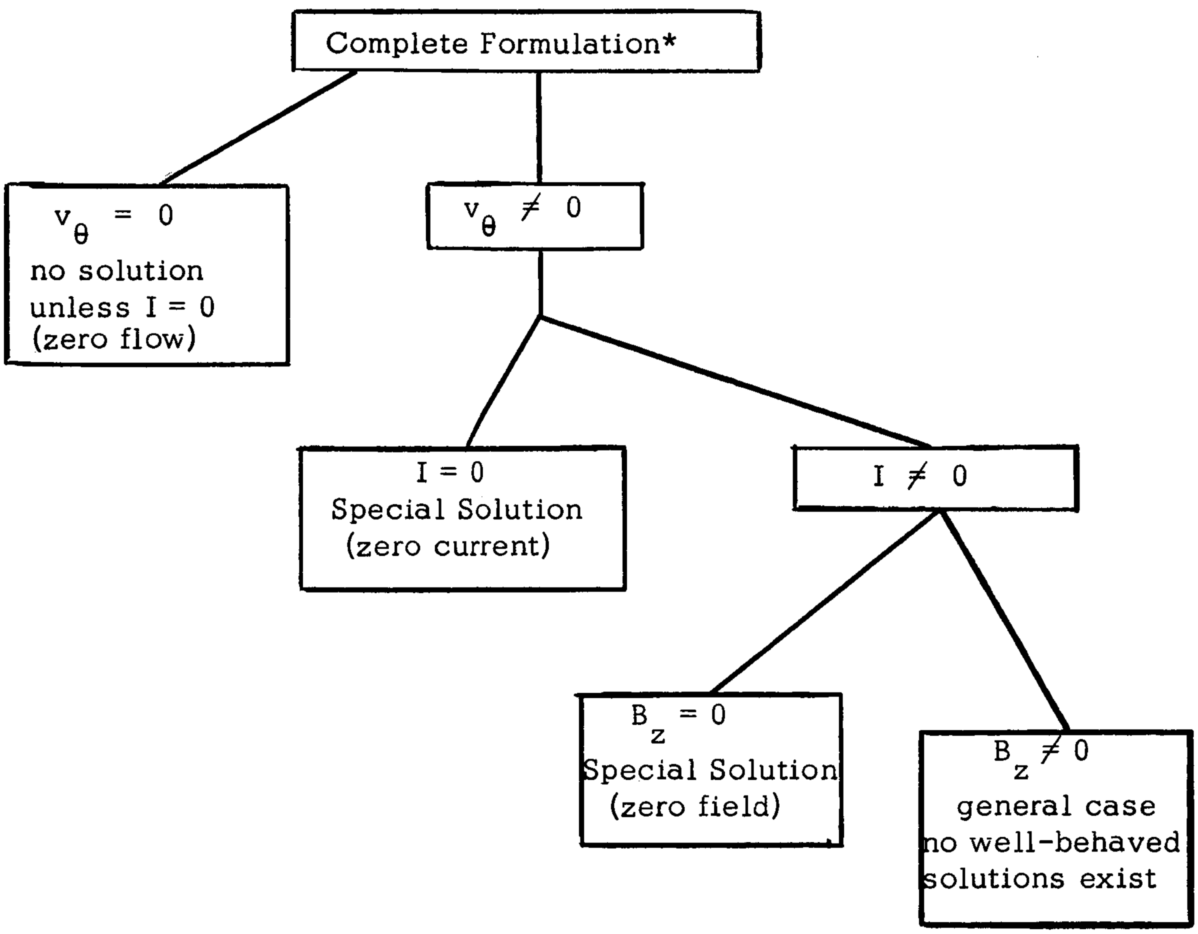
(b) $B_r = 0$ on $z = \pm L$

This is consistent with boundary condition (3). From Eq. (17), if $B_z(z = \pm L) \neq 0$ then $\partial B_\theta/\partial z = 0$ on $z = \pm L$. Yet, since $v_\theta \neq 0$ on $z = \pm L$, boundary condition (1) can be satisfied. Since no contradictions arise here, we conclude

$$B_r = 0 \quad \text{on} \quad z = \pm L. \tag{42}$$

E. Summary

We summarize the analyses given above in the chart below.



*Inviscid, steady, no secondary flow, perfectly conducting electrodes, etc.

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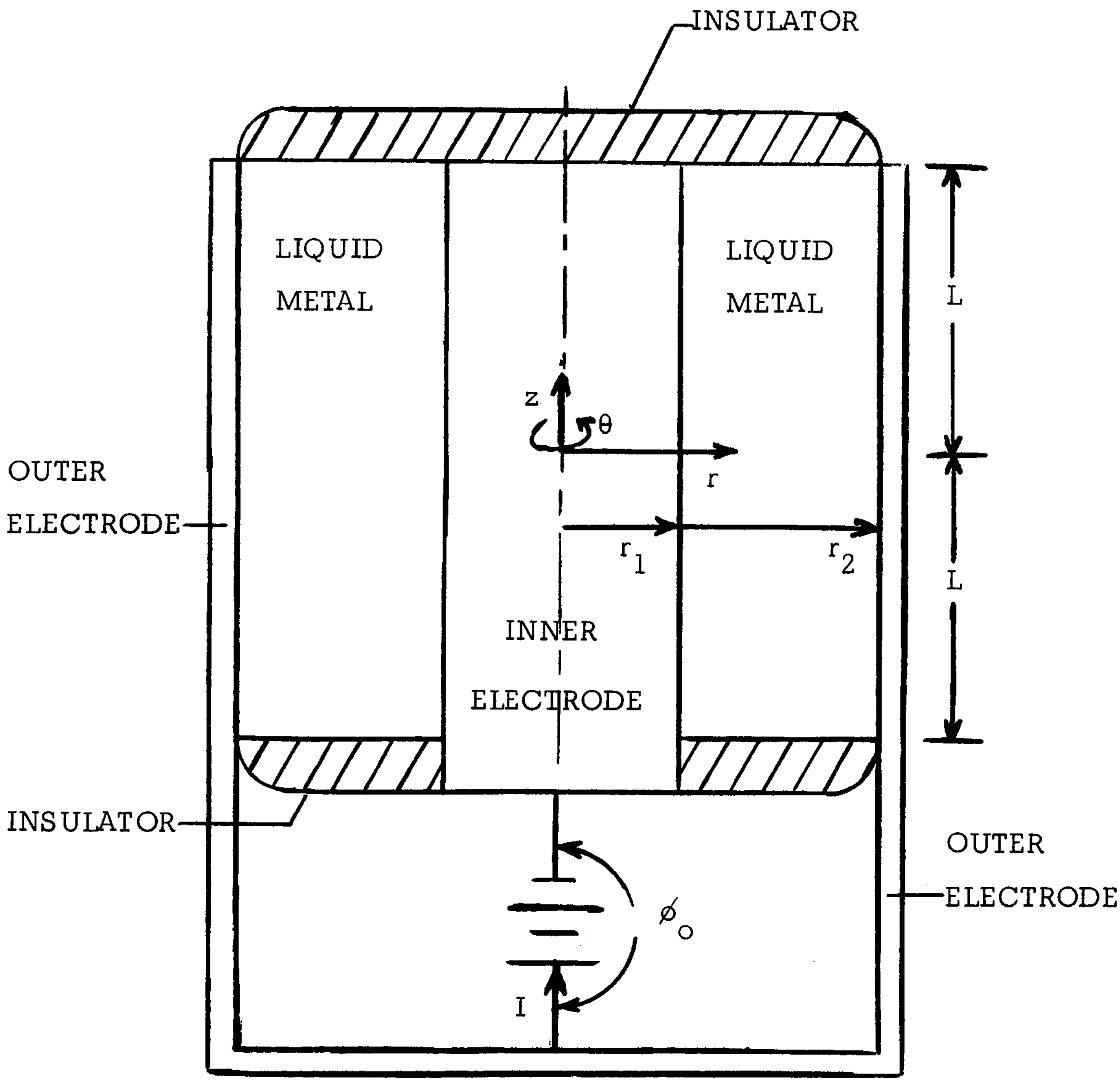
FIGURE CAPTION

Sketch of the annular gap and geometric configuration.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